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Symplectic Geometry and Infinite-Dimensional Symmetry Groups

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Symplectic Geometry and Infinite-Dimensional Symmetry Groups

H. Aratyn[3], E. Nissimov[2] and S. Pacheva[4]

1. Introduction

In this talk we present the basics of the symplectic approach to describe physical models possessing infinite-dimensional symmetries. This approach points the systematic way from the group theoretical objects, through geometric actions, to Poisson bracket structures on infinite-dimensional groups and quantization. Our aim is to uncover the universal features of various important physical theories and obtain new insights by putting them in the framework of coadjoint orbits.

In our formalism each physical model is fully characterized by few fundamental ingredients: pairing $\langle\cdot|\cdot\rangle$ between Lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$, (non-)trivial Lie algebra two-cocycle $\omega(\cdot,\cdot)$ and the adjoint action of the corresponding group G. These ingredients expose fully the symmetry structure of the model and enter into the general recipe for the geometric action. Classical r-matrices and Yang-Baxter equations appear naturally in this geometric setting.

1.1 SO(3) Example: Coadjoint Orbit and Geometric Action

We start illustrating the general formalism with the simple and well-known example of $G = SO(3)$. The interesting geometric structures: Lie-Berezin bracket, Kirillov- Kostant (KK) symplectic form [1], geometric action on the coadjoint orbit and the Poisson brackets induced on the space of functions on G, on which we will comment in greater detail later, will show up already here.

Let $\mathcal{G}$ be the Lie algebra of $SO(3)$ with basis elements e_1, e_2, e_3 and the commutation relations $[e_i\,,\,e_j] = \varepsilon_{ijk}e_k$. A dual basis $\{e^i\}$ in $\mathcal{G}^*$ is defined according to $\langle e^i|e_j\rangle = \delta^i_j$.

A Poisson structure on the space $C^\infty(\mathcal{G}^*, \mathbb{R})$ of smooth, real valued functions on $\mathcal{G}^*$ is given by the Lie-Berezin bracket:

$$\{f(U)\,,\,g(U)\}_{LB} \equiv -\varepsilon_{ijk}u_i\,\partial_j f(U)\,\partial_k g(U) \qquad \forall\, U = u_k e^k \in \mathcal{G}^* \tag{1}$$

It is easy to verify that (1) vanishes on function $f_0(U) = u_i u_i$ invariant under $SO(3)$ rotations (coadjoint action). To remove such degeneracy one restricts the space by imposing $f_0(U) = u_0^2 =$ constant. This restriction defines the orbit of the coadjoint representation of $G = SO(3)$ to be a sphere S^2 with a non-zero radius $u_0 \neq 0$. The inverse of the bracket (1) restricted to the orbit defines the KK symplectic two-form $\hat{\Omega}_U$. The canonical expression for this KK two-form in components is:

$$\hat{\Omega}_U = \frac{1}{2}\omega_{ij}du_i du_j \qquad ; \qquad \omega_{ij} = \varepsilon_{ijk}u_k/u_0^2 \tag{2}$$

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The infinitesimal variation of the co-orbit point U along the orbit (i.e. $u_i du_i = 0$) can be described in terms of the $\mathcal{G}$ valued one-form $Y = y^i e_i$ through $du_k = \varepsilon_{ijk} y^i u_j$. Y must satisfy, for consistency, the Maurer-Cartan (MC) equation $dy^i = \frac{1}{2}\varepsilon_{ijk} y^j y^k$. Using the MC one-form Y we can rewrite (2) in an alternative way as:

$$\hat{\Omega}_U = \frac{1}{2}\langle U|[Y,Y]\rangle/u_0^2 = \frac{1}{2}y^i du_i/u_0^2 \tag{3}$$

In this form the closure of $\hat{\Omega}_U$ becomes transparent due to the Jacobi relation for $so(3)$. In fact $\hat{\Omega}_U$ is exact and is reproduced by the Liouville one-form α_u through

$$\hat{\Omega}_U = d\alpha_u \qquad \text{with} \qquad \alpha_u \equiv -u_k y^k \tag{4}$$

Let us now parametrize the orbit S^2 by spherical coordinates:

$$u_1 = u_0 \sin\theta\cos\phi \quad , \quad u_2 = u_0 \sin\theta \sin\phi \quad ; \quad u_3 = u_0 \cos\theta \tag{5}$$

Using spherical coordinates one can easily rewrite (2) in terms of Darboux variables ϕ and $\cos\theta$ as

$$\hat{\Omega}_U = u_0\, d\phi\, d\cos\theta \tag{6}$$

and find the MC one-form to be

$$y^1(\theta,\phi) = -\sin\phi\, d\theta \;\; ; \;\; y^2(\theta,\phi) = \cos\phi\, d\theta \;\; ; \;\; y^3(\theta,\phi) = d\phi \tag{7}$$

where y^i is defined up to an expression βu_i with β being an arbitrary closed one-form (this gauge ambiguity in defining MC one-form $Y = y^i e_i$ will not affect $\hat{\Omega}_U$ given by (4)). Clearly (7) satisfies the MC equation.

Under infinitesimal $SO(3)$ transformation:

$$U \to U' = U + [\epsilon, U] \quad \text{and} \quad Y \to Y' = Y + [\epsilon, Y] + d\epsilon \quad \text{for} \quad \epsilon = \epsilon^i e_i \tag{8}$$

Correspondingly, the Liouville one-form $\alpha_u = -\langle U|Y\rangle$ is not invariant under $SO(3)$ transformation but transforms as $\alpha_u \to \alpha_u - \langle U|d\epsilon\rangle$. If we now define the geometric action for $SO(3)$ as

$$W \equiv \int \alpha_u = -\int \langle U \mid Y\rangle = -u_0 \int \cos\theta\; d\phi \tag{9}$$

we find that (8) results in the transformation rule $\delta W = -\int\langle U|d\epsilon\rangle$. Hence the Noether procedure for the geometric action (9) identifies the co-orbit point U as a conserved momentum (see eq.(41) below for generalization of this important feature).

1.2 SO(3) Example: Poisson Brackets on the Group and Classical r-matrix

The orbit of $G = SO(3)$ is naturally embedded in the larger phase space given by the cotangent bundle T^*G equipped with the canonical symplectic two-form :

$$\Omega(U,g) = -d\,\langle\, U \mid dgg^{-1}\,\rangle \tag{10}$$

where the canonical momentum $U = u_k e^k$ belongs to the cotangent space T_g^*G at the point $g \in G$ and T_g^*G can be identified with the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of G. Note, that despite of the similarity between (4) and (10), these objects are defined on different spaces.

The non-zero Poisson brackets (PB's) among the canonical momenta and coordinates (U, g) corresponding to the symplectic two-form (10) are:

$$\begin{aligned} \{u^i\,,\,u^j\}_{PB} &= -\varepsilon_{ijk}u^k \\ \{u^i\,,\,\Phi(g)\}_{PB} &= L_i\Phi(g) \equiv \frac{d}{dt}\Phi(e^{te_i}g)\Big|_{t=0} \qquad \text{for} \quad \Phi \in C^\infty(G, \mathrm{I\!R}) \end{aligned} \tag{11}$$

Consider now a reduction of T^*G to the orbit passing through the point U_0 by imposing:

$$\Psi_\xi(U, g) \equiv \left\langle g^{-1}Ug - U_0 \mid \xi \right\rangle = 0 \tag{12}$$

The constraints (12) satisfy the PB algebra:

$$\{\,\Psi_\xi(U,g)\,,\,\Psi_\eta(U,g)\,\}_{PB} = \Psi_{[\xi\,,\,\eta]}(U,g) + \langle U_0 \mid [\xi\,,\,\eta]\,\rangle \tag{13}$$

The orbit passing through U_0 is isomorphic to G/G_{stat} where G_{stat} is the stationary subgroup of the point U_0 corresponding to the Lie subalgebra

$$\mathcal{G}_{stat} \equiv \{\xi_0 = \xi_0^i e_i \in \mathcal{G}\ ;\ \varepsilon_{ijk}\,\xi_0^i\,u_{0\,j}e^k = 0\} \tag{14}$$

The second class part of constraints (12) is given by $\Psi_\perp \equiv \Psi_{\xi_\perp}(U,g)$ with $\xi_\perp \in \mathcal{G} \setminus \mathcal{G}_{stat}$. We find the Dirac brackets (DB's) between smooth functions $\Phi_{1,2}(g)$ on the reduced phase space G/G_{stat} to be:

$$\begin{aligned} \{\Phi_1(g), \Phi_2(g)\}_{DB} &= -\left\{\Phi_1(g), \Psi_\perp^i(U,g)\right\}_{PB} \left(\langle U_0 \mid [e_i\,,\,e_j]\rangle\right)^{-1} \left\{\Psi_\perp^j(U,g), \Phi_2(g)\right\}_{PB} \\ &= -r_{ij}R^i\Phi_1(g)R^j\Phi_2(g) \end{aligned} \tag{15}$$

where

$$R^i\Phi(g) = \frac{d}{dt}\Phi\left(ge^{te_i}\right)\Big|_{t=0} \tag{16}$$

denotes the right Lie derivative along e_i. The r-matrix entering (15) has indices i, j running outside $\mathcal{G}_{stat}$ where it is defined by:

$$r^{-1}_{ij} = -\varepsilon_{ijk}u_{0\,k} \qquad ; \qquad r_{ij} = \varepsilon_{ijk}u_{0\,k}/u_0^2 \tag{17}$$

Let us introduce the matrix $r \equiv r_{ij}\,e_i \otimes e_j \in \mathcal{G} \otimes \mathcal{G}$ for which we find an explicit formula

$$r = \frac{1}{u_0^2}\,e_i \otimes (\vec{e} \times \vec{u_0})_i \quad \to \quad r = \frac{1}{u_0}(e_1 \otimes e_2 - e_2 \otimes e_1) \quad \text{for} \quad u_{0\,i} = \delta_{i3}u_0 \tag{18}$$

With the usual notation $r^{(12)} \equiv r_{ij}e_i \otimes e_j \otimes 1\!\!1$ we can rewrite the Jacobi identity of the Dirac bracket (15) in the form of the well-known classical Yang-Baxter equation

$$[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] + [r^{(13)}, r^{(23)}] = 0 \quad \text{with} \quad r^{(12)} = -r^{(21)} \tag{19}$$

See ref.[13] for generalization of the classical r-matrix construction (15) to infinite-dimensional groups. In section 4 below we discuss application to Virasoro group.

2. General Formalism

2.1 Basic Ingredients

Let us now consider arbitrary (infinite-dimensional) group G with a Lie algebra $\mathcal{G}$ and its dual space $\mathcal{G}^*$. The adjoint and coadjoint actions of G and $\mathcal{G}$ on $\mathcal{G}$ and $\mathcal{G}^*$ are given by $Ad(g)(\xi) = g\xi g^{-1}$, $ad(\xi)\eta = [\xi, \eta]$ and $\langle Ad^*(g)U|\xi\rangle = \langle U|Ad(g^{-1})\xi\rangle$, $\langle ad^*(\xi)U|\eta\rangle = -\langle U|ad(\xi)\eta\rangle$. Here $g \in G$ and $\xi, \eta \in \mathcal{G}$, $U \in \mathcal{G}^*$ are arbitrary elements, whereas $\langle\cdot|\cdot\rangle$ indicates the natural bilinear form "pairing" $\mathcal{G}$ and $\mathcal{G}^*$.

Our primary interest is in infinite-dimensional Lie algebras with a central extension $\widetilde{\mathcal{G}} = \mathcal{G} \oplus \mathbb{R}$ of $\mathcal{G}$ and, correspondingly, an extension $\widetilde{\mathcal{G}}^* = \mathcal{G}^* \oplus \mathbb{R}$ of the dual space $\mathcal{G}^*$. The central extension is given by a linear operator $\hat{s} : \mathcal{G} \longrightarrow \mathcal{G}^*$ satisfying

$$\hat{s}([\xi, \eta]) = ad^*(\xi)\hat{s}(\eta) - ad^*(\eta)\hat{s}(\xi) \tag{20}$$

which defines a nontrivial two-cocycle on the Lie algebra $\mathcal{G}$:

$$\omega(\xi, \eta) \equiv -\lambda\, \langle\hat{s}(\xi) \mid \eta\rangle \qquad\qquad \forall\, \xi, \eta \in \mathcal{G} \tag{21}$$

where λ is a numerical normalization constant. The Jacobi identity (20) can be integrated ($\eta \longrightarrow g = \exp\eta$) to get a unique nontrivial $\mathcal{G}^*$-valued group one-cocycle $S(g)$ in terms of the Lie-algebra cocycle operator $\hat{s}$ (provided $H^1(G) = \emptyset$, $\dim H^2(G) = 1$; see [2]) :

$$ad^*(\xi)S(g) = Ad^*(g)\,\hat{s}\left(Ad(g^{-1})\xi\right) - \hat{s}(\xi) \qquad\qquad \forall\, \xi \in \mathcal{G} \tag{22}$$

satisfying the relations :

$$\hat{s}(\xi) = \frac{d}{dt}S(e^{t\xi})\Bigg|_{t=0} \quad , \quad S(g_1 g_2) = S(g_1) + Ad^*(g_1)S(g_2) \tag{23}$$

Now, we can easily generalize the adjoint and coadjoint actions of G and $\mathcal{G}$ to the case with a central extension (acting on elements $(\xi, n), (\eta, m) \in \widetilde{\mathcal{G}}$ and $(U, c) \in \widetilde{\mathcal{G}}^*$; see e.g. [3])[5] :

$$\widetilde{Ad}(g)\,(\xi, n) \;=\; \left(Ad(g)\xi\;,\; n + \lambda\left\langle S(g^{-1})\,|\,\xi\right\rangle\right) \tag{24}$$

$$\widetilde{ad}(\xi, n)\;(\eta, m) \;\equiv\; [(\xi, n)\,,\,(\eta, m)] = \left(ad(\xi)\;\eta\;,\; -\lambda\,\langle\hat{s}(\xi)|\eta\rangle\right) \tag{25}$$

$$\widetilde{Ad}^*(g)\,(U, c) = (Ad^*(g)U + c\lambda\, S(g)\;,\; c) \quad , \quad \widetilde{ad}^*(\xi, n)(U, c) = (ad^*(\xi)\,U + \,c\,\lambda\,\hat{s}(\xi)\;,\;0\,) \tag{26}$$

Also, the bilinear form $\langle\cdot|\cdot\rangle$ on $\mathcal{G}^* \otimes \mathcal{G}$ can be extended to a bilinear form on $\widetilde{\mathcal{G}}^* \otimes \widetilde{\mathcal{G}}$ as :

$$\langle(U, c) \mid (\xi, n)\rangle = \langle U \mid \xi\rangle + c\,n \tag{27}$$

Another basic geometric object is the fundamental $\mathcal{G}$-valued Maurer-Cartan one-form $Y(g)$ on G satisfying $dY(g) = \frac{1}{2}\left[\,Y(g)\,,\,Y(g)\,\right]$. It is related to the group one-cocycle $S(g)$ through the equation :

$$d\,S(g) = ad^*(Y(g))\,S(g) + \hat{s}(Y(g)) \tag{28}$$

[5] The physical interpretation of the $\mathcal{G}$-cocycle $\hat{s}$ is that of "anomaly" of the Lie algebra (i.e., existence of a c-number term in the commutator (25)), whereas the group cocycle $S(g)$ is the integrated "anomaly", i.e. the "anomaly" for finite group transformations (see eqs.(24) and (23)).

and possesses group one-cocycle property similar to that of $S(g)$ (23) :

$$Y(g_1 g_2) = Y(g_1) + Ad(g_1) Y(g_2) \tag{29}$$

The group- and algebra-cocycles $S(g)$ and $\hat{s}(\xi)$ can be generalized to include trivial (co-boundary) parts ((U_0, c) being an arbitrary point in the extended dual space $\widetilde{\mathcal{G}}^*$) :

$$\Sigma(g) \equiv \Sigma\Bigl(g; (U_0, c)\Bigr) = c\lambda S(g) + Ad^*(g) U_0 - U_0 \tag{30}$$

$$\hat{\sigma}(\xi) \equiv \hat{\sigma}(\xi; (U_0, c)) = ad^*(\xi) U_0 + c\lambda\, \hat{s}(\xi) = \frac{d}{dt} \Sigma(e^{t\xi}) \Bigg|_{t=0} \tag{31}$$

The generalized cocycles (30) and (31) satisfy the same relations as (23), (28) and (22).

2.2 Coadjoint Orbits

The coadjoint orbit of G, passing through the point (U_0, c) of the dual space $\widetilde{\mathcal{G}}^*$, is defined as (cf. (26)) :

$$\mathcal{O}_{(U_0,c)} \equiv \Bigl\{ \Bigl(U(g), c\Bigr) \in \widetilde{\mathcal{G}}^* \;;\; U(g) = U_0 + \Sigma(g) = Ad^*(g) U_0 + c\lambda S(g) \Bigr\} \tag{32}$$

The orbit (32) is a right coset $\mathcal{O}_{(U_0,c)} \simeq G/G_{stat}$ where G_{stat} is the stationary subgroup of the point (U_0, c) w.r.t. the coadjoint action (26) :

$$G_{stat} = \Bigl\{ k \in G \;;\; \Sigma(k) \equiv c\lambda S(k) + Ad^*(k) U_0 - U_0 = 0 \Bigr\} \tag{33}$$

The Lie algebra corresponding to G_{stat} is :

$$\mathcal{G}_{stat} \equiv \Bigl\{ \xi_0 \in \mathcal{G} \;;\; \hat{\sigma}(\xi_0) \equiv ad^*(\xi_0) U_0 + c\lambda\, \hat{s}(\xi_0) = 0 \Bigr\} \tag{34}$$

Now, using the basic geometric objects from sect. 2.1, we can express the KK symplectic form Ω_{KK} [1] on $\mathcal{O}_{(U_0,c)}$ for any infinite-dimensional (centrally extended) group G in a simple compact form [3] . Namely, introducing the centrally extended objects :

$$\widetilde{\Sigma}(g) \equiv (\Sigma(g), c) \in \widetilde{\mathcal{G}}^* \quad , \quad \widetilde{Y}(g) \equiv \Bigl(Y(g), m_Y(g)\Bigr) \in \widetilde{\mathcal{G}} \tag{35}$$

$$d\widetilde{\Sigma}(g) = \widetilde{a}d^*\Bigl(\widetilde{Y}(g)\Bigr)\widetilde{\Sigma}(g) \quad , \quad d\widetilde{Y}(g) = \frac{1}{2}\Bigl[\widetilde{Y}(g)\, ,\, \widetilde{Y}(g)\Bigr] \tag{36}$$

we obtain (using (27) and (28)) :

$$\Omega_{KK} = -d\Bigl(\Bigl\langle \widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \Bigr\rangle\Bigr) = -\frac{1}{2}\Bigl\langle d\widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \Bigr\rangle \tag{37}$$

2.3 Geometric Actions, Symmetries and Ward Identities

The geometric action on a coadjoint orbit $\mathcal{O}_{(U_0,c)}$ of arbitrary infinite-dimensional (centrally extended) group G can now be written down compactly as [3, 4] :

$$W[g] = \int d^{-1}\Omega_{KK} = -\int \Bigl\langle \widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \Bigr\rangle \tag{38}$$

or, in more detail, introducing the explicit expressions (35), (36), (30) and (31) :

$$W[g] = \int \Big\langle U_0 \mid Y(g^{-1}) \Big\rangle - c\lambda \int \Big[\Big\langle S(g) \mid Y(g) \Big\rangle - \frac{1}{2} d^{-1} \Big(\Big\langle \hat{s}(Y(g)) \mid Y(g) \Big\rangle \Big) \Big] \tag{39}$$

The integral in (38), (39) is over one-dimensional curve on the phase space $\mathcal{O}_{(U_0,c)}$ with a "time-evolution" parameter τ . Along the curve the exterior derivative becomes $d = d\tau\partial_\tau$ and the projection of the one-form $Y(g)$ is : $Y(g) = d\tau y_\tau(g)$.

The fundamental Poisson brackets $\Big\{ \Big\langle \widetilde{\Sigma}(g) \mid \widetilde{\xi} \Big\rangle \, , \, \Big\langle \widetilde{\Sigma}(g) \mid \widetilde{\eta} \Big\rangle \Big\}_{PB} = - \Big\langle \widetilde{\Sigma}(g) \mid [\widetilde{\xi} \, , \, \widetilde{\eta}] \Big\rangle$ arising from (39) identify $\widetilde{\Sigma}(g)$ as an equivariant moment map.

The group cocycle properties of $S(g)$ and $Y(g)$ (eqs.(23) and (29)) imply the following fundamental group composition law [4] (with $\Sigma(g)$ as in (30)) :

$$W[g_1 g_2] = W[g_1] + W[g_2] + \int \Big\langle \Sigma(g_2) \mid Y(g_1^{-1}) \Big\rangle \tag{40}$$

Eq.(40) is a generalization of the famous Polyakov-Wiegmann composition law [5] in WZNW models to geometric actions on coadjoint orbits of arbitrary groups with central extensions.

Eq.(40) contains the whole information about the symmetries of the geometric action (39). First, under arbitrary left group translations $g \longrightarrow \exp t\xi \, g$ we obtain, using (40), the Noether theorem :

$$\frac{d}{dt} W[\exp t\xi \, g] \Big|_{t=0} \equiv L_\xi W[g] = - \int \langle \Sigma(g) \mid d\xi \rangle \tag{41}$$

i.e., $\Sigma(g)$ (30) is a Noether conserved current : $\partial_\tau \Sigma(g) = 0$.

Next, under arbitrary right group translations $g \longrightarrow g \exp t\eta$ we get from (40) :

$$\frac{d}{dt} W[g \, \exp t\xi] \Big|_{t=0} \equiv R_\eta W[g] = \int \Big\langle \hat{\sigma}(\eta) \mid Y(g^{-1}) \Big\rangle = - \int \Big\langle \hat{\sigma}(Y(g^{-1})) \mid \eta \Big\rangle \tag{42}$$

Recalling (34) we find "gauge" invariance of $W[g]$ under right group translations from the stationary subgroup G_{stat} (33) of the orbit $\mathcal{O}_{(U_0,c)}$ (32) : $R_{\xi_0} W[g] = 0$ for $\forall \xi_0 \in \mathcal{G}_{stat}$ (34). This reveals the geometric meaning of "hidden" local symmetries [6] in models with arbitrary infinite-dimensional Noether symmetry groups.

The Legendre transform of $W[g]$:

$$\Gamma[g] \equiv W[g] - \int \Big\langle \Sigma(g) \mid \frac{\delta W[g]}{\delta \Sigma(g)} \Big\rangle = W[g] + \int \langle \Sigma(g) \mid Y(g) \rangle = -W[g^{-1}] \tag{43}$$

considered as a functional of $Y(g) = d\tau y_\tau(g)$, satisfies the functional equation :

$$\partial_\tau \frac{\delta \Gamma}{\delta y_\tau(g)} - ad^*(y_\tau(g)) \frac{\delta \Gamma}{\delta y_\tau(g)} - \hat{s}(y_\tau(g)) = 0 \tag{44}$$

As shown in [4, 7], eq.(44) coincides with the Ward identity for the functional integral :

$$\exp i\Gamma[y] = \int \mathcal{D}h \exp i \Big\{ W[h] + \int \langle \Sigma(h) \mid y \rangle \Big\} \tag{45}$$

and thus (43) provides the exact solution of (45) upon parametrizing $y = y_\tau(g)$.

3. Applications

3.1 Kac-Moody Groups

The Kac-Moody group elements $g \simeq g(x)$ are smooth mappings $S^1 \longrightarrow G_0$, where G_0 is a finite-dimensional Lie group with generators $\{T^A\}$. The explicit form of (24)-(26) reads in this case :

$$Ad(g)\xi = g(x)\xi(x)g^{-1}(x) \quad , \quad ad(\xi_1)\xi_2 = [\xi_1(x)\,,\,\xi_2(x)] \quad , \quad \xi_{1,2}(x) = \xi^A_{1,2}(x)T_A$$
$$Ad^*(g)U = g(x)U(x)g^{-1}(x) \quad , \quad ad^*(\xi)U = [\xi(x)\,,\,U(x)] \quad , \quad U(x) = U_A(x)T^A$$
$$\hat{s}(\xi) = \partial_x\xi(x) \quad , \quad S(g) = \partial_x g(x)\,g^{-1}(x) \quad , \quad Y(g) = dg(x)\,g^{-1}(x) \tag{46}$$

Plugging (46) into (39) one obtains the well-known WZNW action for G_0-valued chiral fields coupled to an external "potential" $U_0(x)$.

3.2 Virasoro Group

The Virasoro group elements $g \simeq F(x)$ are smooth diffeomorphisms of the circle S^1 . Group multiplication is given by composition of diffeomorphisms in inverse order : $g_1 \cdot g_2 = F_2 \circ F_1\;(x) = F_2\Bigl(F_1(x)\Bigr)$. Eqs.(24)-(26) have now the following explicit form :

$$Ad(F)\xi = \Bigl(\partial_x F\Bigr)^{-1}\xi\,(F(x)) \quad , \quad Ad^*(F)U = \Bigl(\partial_x F\Bigr)^2 U\,(F(x))$$
$$ad(\xi)\eta \equiv [\xi,\eta] = \xi\partial_x\eta - (\partial_x\xi)\eta \quad , \quad ad^*(\xi)U = \xi\partial_x U + 2(\partial_x\xi)U$$
$$\hat{s}(\xi) = \partial_x^3\xi \quad , \quad S(F) = \frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2}\left(\frac{\partial_x^2 F}{\partial_x F}\right)^2 \quad , \quad Y(F) = \frac{dF}{\partial_x F} \tag{47}$$

Here $S(F)$ is the well-known Schwarzian. Plugging (47) into the general expressions (39) and (40) one reproduces the well-known Polyakov $D = 2$ gravity action (coupled to an external stress-tensor $U_0(x)$) :

$$W[F] = \int d\tau dx \left[-U_0(F(\tau,x))\,\partial_x F\,\partial_\tau F + \frac{c}{48\pi}\frac{\partial_\tau F}{\partial_x F}\left(\frac{\partial_x^3 F}{\partial_x F} - 2\frac{(\partial_x^2 F)^2}{(\partial_x F)^2}\right)\right] \tag{48}$$

and its group composition law [6, 8].

3.3 $(\mathbf{N},\mathbf{0})$ $\mathbf{D=2}$ Super-Virasoro Group $(\mathbf{N} \leq \mathbf{4})$

Here we shall use the manifestly $(N,0)$ supersymmetric formalism. The points of the $(N,0)$ superspace are labeled as (t,z), $z \equiv (x,\theta^i)$, $i = 1,..,N$. The group elements are given by superconformal diffeomorphisms :

$$z \equiv (x,\theta^j) \;\longrightarrow\; \widetilde{Z} \equiv \Bigl(F(x,\theta^j), \widetilde{\Theta}^i(x,\theta^j)\Bigr) \tag{49}$$

obeying the superconformal constraints [6] :

$$D^j F - i\widetilde{\Theta}^k D^j\widetilde{\Theta}_k = 0 \quad , \quad D^j\widetilde{\Theta}^l D^k\widetilde{\Theta}_l - \delta^{jk}\Bigl[D\widetilde{\Theta}\Bigr]^2_N = 0 \quad , \quad \Bigl[D\widetilde{\Theta}\Bigr]^2_N \equiv \frac{1}{N}D^m\widetilde{\Theta}^n D_m\widetilde{\Theta}_n \tag{50}$$

[6] The following superspace notations are used : $D^i = \frac{\partial}{\partial\theta_i} + i\theta^i\partial_x$, $D^N \equiv \frac{1}{N!}\epsilon_{i_1\cdots i_N}D^{i_1}\cdots D^{i_N}$.

The $(N,0)$ supersymmetric analogues of (47) read:

$$Ad(\widetilde{Z})\xi = \left(\left[D\widetilde{\Theta}\right]^2_N\right)^{-1}\xi(\widetilde{Z}(z)) \quad , \quad Ad^*(\widetilde{Z})U = \left(\left[D\widetilde{\Theta}\right]^2_N\right)^{2-\frac{N}{2}} U(\widetilde{Z}(z)) \tag{51}$$

$$ad(\xi)\eta \equiv [\xi,\eta] = \xi\partial_x\eta - (\partial_x\xi)\eta - \frac{i}{2}D_k\xi D^k\eta \quad , \quad ad^*(\xi)U = \xi\partial_x U + (2-\frac{N}{2})(\partial_x\xi)U - \frac{i}{2}D_k\xi D^k U \tag{52}$$

$$\hat{s}_N(\xi) = i^{N(N-2)}D^N\partial_x^{3-N}\xi \quad , \quad Y_N(\widetilde{Z}) = \left(dF + i\widetilde{\Theta}^j d\widetilde{\Theta}_j\right)\left(\left[D\widetilde{\Theta}\right]^2_N\right)^{-1} \tag{53}$$

The associated $\mathcal{G}^*$-valued group one-cocycles $S_N(\widetilde{Z})$ coincide with the well-known [9] $(N,0)$ super-Schwarzians. Inserting the latter and (53) into (39) one obtains the $(N,0)$ supersymmetric generalization of the Polyakov $D=2$ gravity action for any $N \leq 4$:

$$W_N[\widetilde{Z}] = \int d\tau\;(dz)\left[\partial_\tau\left(\ln\left[D\widetilde{\Theta}\right]^2_N\right)D^N\partial_x^{1-N}\left(\left[D\widetilde{\Theta}\right]^2_N\right) - U_0(\widetilde{Z})\left(\left[D\widetilde{\Theta}\right]^2_N\right)^{2-\frac{N}{2}}Y_N(\widetilde{Z})\right] \tag{54}$$

3.4 Group of Area-Preserving Diffeomorphisms on Torus with Central Extension $\widetilde{\mathbf{SDiff}}(\mathbf{T^2})$

The elements of $\widetilde{\mathrm{SDiff}}\,(T^2)$ are described by smooth diffeomorphisms $T^2 \ni \vec{x} \equiv (x^1,x^2) \longrightarrow F^i(\vec{x}) \in T^2$ $(i=1,2)$, such that $\det \parallel \frac{\partial F^i}{\partial x^j} \parallel = 1$. The Lie algebra of $\widetilde{\mathrm{SDiff}}\,(T^2)$ reads : $\left[\hat{\mathcal{L}}(\vec{x}),\hat{\mathcal{L}}(\vec{y})\right] = -\epsilon^{ij}\partial_i\hat{\mathcal{L}}(\vec{x})\partial_j\delta^{(2)}(\vec{x}-\vec{y}) - a^i\partial_i\delta^{(2)}(\vec{x}-\vec{y})$, where $\vec{a} \equiv (a^1,a^2)$ are the "central charges" [10] . The general eqs.(24)-(26) now specialize to [11] :

$$Ad(\vec{F})\xi = \xi(\vec{F}(\vec{x})) \quad , \quad ad(\xi)\eta \equiv [\xi,\eta](\vec{x}) = \epsilon^{ij}\partial_i\xi(\vec{x})\partial_j\eta(\vec{x})$$
$$Ad^*(\vec{F})U = U(\vec{F}(\vec{x})) \quad , \quad ad^*(\xi)U = \epsilon^{ij}\partial_i\xi(\vec{x})\partial_j U(\vec{x})$$

$$\hat{s}(\xi) = a^i\partial_i\xi(\vec{x}) \quad , \quad S(\vec{F}) = a^i\epsilon_{ij}\left(F^j(\vec{x}) - x^j\right) \quad , \quad Y(\vec{F}) = \frac{1}{2}\epsilon_{ij}F^i dF^j + d\rho(\vec{F}) \tag{55}$$

where $\partial_i\rho(\vec{F}) = -\frac{1}{2}\left(\epsilon_{kl}F^k\partial_i F^l + \epsilon_{ij}x^j\right)$.

Plugging (55) into (39) we get the $\widetilde{\mathrm{SDiff}}\,(T^2)$ co-orbit geometric action [11] :

$$W_{\widetilde{\mathrm{SDiff}}\,(T^2)}[\vec{F}] = -\frac{1}{3}\int d\tau\,dx^2\left(a^k\epsilon_{kl}F^l\right)\epsilon_{ij}F^i\partial_\tau F^j \tag{56}$$

In [7] it was shown that (56) is the Wess-Zumino anomalous effective action for the toroidal membrane in the light-cone gauge.

3.5 Gel'fand-Dikii Algebra (GDA) and $\mathbf{W_n}$ Symmetry

The $SL(n)$ GDA consists of differential operators $L(u) = \sum_{k=0}^n u_k(x)\partial_x^k$ with $u_n = 1$, $u_{n-1} = 0$. Its dual space is the Volterra algebra of pseudo-differential operators $\hat{\xi} = \sum_{k\geq 0}\partial_x^{-k-1}(\xi_k(x)\cdot)$ with $\left\langle L(u)\mid\hat{\xi}\right\rangle = \int dx\,\mathrm{res}\left(L(u)\,\hat{\xi}\right) = \int dx\sum_{k=0}^n u_k(x)\,\xi_k(x)$ defining the bilinear form. On GDA there exist natural Poisson bracket structures known as first- and second- GD brackets [12].

Let us introduce a Maurer-Cartan-like one-form with values in the Volterra algebra $\hat{Y} = \sum_{k=0}^{n} \partial_x^{-k}(Y_k(x)\cdot)$ satisfying [7] (cf. first eq.(36)) :

$$dL(u) = \Bigl(L(u)\hat{Y}\Bigr)_+ L(u) - L(u)\Bigl(\hat{Y}L(u)\Bigr)_+ \tag{57}$$

The r.h.s. of (57) represents the coadjoint action of Volterra algebra on GDA. Now, we can write compactly the symplectic two-form $\Omega^{(2)}_{GD}$, corresponding to the second GD bracket, and the associated geometric action, describing models with W_n symmetry, as (cf. (37) and (38)) :

$$W_n[u] = \int d^{-1}\Omega^{(2)}_{GD} = -\frac{1}{2}\int d^{-1}\Bigl(\Bigl\langle dL(u) \mid \hat{Y}(u)\Bigr\rangle\Bigr) \tag{58}$$

where $\hat{Y}(u)$ is the solution of (57).

4. Poisson Brackets and Classical r-matrix for Virasoro Group

We find the infinite-dimensional analogue of (15) for Virasoro group to be :

$$\{F(x), F(y)\}_{DB} = -r(F(x), F(y); \{e\}) = -\frac{1}{2c\lambda}\Bigl\{\frac{1}{\mu^2}\varepsilon(x-y)\Bigl[1-\cos\mu\Bigl(F(x)-F(y)\Bigr)\Bigr]$$
$$+ \quad e_1 \sin\mu\Bigl(F(x)-F(y)\Bigr) + e_2\Bigl[\sin\mu F(x) - \sin\mu F(y)\Bigr] + e_3\Bigl[\cos\mu F(x) - \cos\mu F(y)\Bigr]\Bigr\} \tag{59}$$

Notations in (59) are as follows. The normalization constant λ is $-1/24\pi$. $\mu^2 \equiv 2U_0/c\lambda$ where the constant U_0 parametrizes a generic Virasoro coadjoint orbit [14]. $r(x,y;\{e\})$ is the operator kernel of the infinite-dimensional Virasoro r-matrix satisfying :

$$c\lambda\partial_x\Bigl(\partial_x^2+\mu^2\Bigr)r(x,y;\{e\}) = \delta(x-y) \quad , \quad r(x,y;\{e\}) = -r(y,x;\{e\})$$
$$\sum_{cyclic\,(1,2,3)} \Bigl[r(x_1,x_2;\{e\})\partial_{x_2}r(x_2,x_3;\{e\}) - \partial_{x_2}r(x_1,x_2;\{e\})\, r(x_2,x_3;\{e\})\Bigr] = 0 \tag{60}$$

Eq. (60) is the differential classical Yang-Baxter equation for Virasoro group. $\{e\} \equiv (e_1,e_2,e_3)$ are constants constrained by $e_2^2+e_3^2-e_1^2 = 1/\mu^4$. The parametric dependence of $r(x,y;\{e\})$ on $\{e\}$ results from the "hidden" $SL(2;\mathbb{R})$ gauge invariance of the Polyakov action (48). To exhibit it more explicitly let us consider, for simplicity, the case of $U_0 = 0$ co-orbit. Then [13] :

$$r(x,y;A_0) = \frac{1}{4c\lambda}\Bigl[(x-y)^2\varepsilon(x-y)+(1+2b_0c_0)(x^2-y^2)-2a_0b_0(x-y)-2c_0d_0xy(x-y)\Bigr] \tag{61}$$

where the coefficients are parametrized in terms of the $SL(2;\mathbb{R})$ matrix $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. It is easy to show that under $SL(2;\mathbb{R})$ transformation $A : F \longrightarrow F\circ A \equiv (aF+b)(cF+d)^{-1}$ the Virasoro group Poisson brackets (59) are $SL(2;\mathbb{R})$-covariant :

$$\{F\circ A\,(x)\,,\ F\circ A\,(y)\}_{DB} = -r\Bigl(F\circ A\,(x)\,,\ F\circ A\,(y)\,;\ AA_0\Bigr) \tag{62}$$

[7] For consistency, the coefficient (one-form) $Y_n(x)$ is determined by the requirement $\mathrm{res}[L(u),\hat{Y}] = 0$. The subscript $+$ indicates taking the differential part.

The above construction extends to other infinite-dimensional groups [13]. The r-operator is an inverse to the Lie-algebra cocycle operator $\hat{\sigma}(\cdot)$ (31) and satisfies a differential Yang-Baxter equation as a consequence of the cocycle condition (20) (cf.(60)). The remaining ambiguity in the r-operator is parametrized by elements of G_{stat} (33).

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